Q2 (a) If u = $x\phi(y/x)+\psi(y/x)$**, prove that** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$

Answer

(2) (a). Let $u = z_1 + z_2$
where $z_1 = x\,\phi(y/x)$ and $z_2 = \psi(y/x)$

Here z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree zero. Therefore, by Euler's theorem —

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(z_1+z_2)+y\frac{\partial}{\partial y}(z_1+z_2)$$
$$= \left(x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right)+\left(x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right)$$
$$= 1.z_1 + 0.z_2 \text{, by Euler's theorem}$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = z_1 \longrightarrow (1)$$

Differentiating (1) w.r.t. x and y respectively, we get —

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial z_1}{\partial x} \longrightarrow (2)$$

and

$$x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = \frac{\partial z_1}{\partial y} \longrightarrow (3)$$

Multiplying (2) by x and (3) by y and adding, we get —

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}$$

or

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+z_1 = 1.z_1$$

$\left[\because x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=z_1\right.$, by (1)
and $\left. x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}=1.z_1\text{, by Euler's theorem}\right]$

or

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

(Proved)

Q2 (b) Show that the maximum and minimum of the radii vectors of the sections of the surface $(x^2+y^2+z^2)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ by the plane $\lambda x+\mu y+\gamma z=0$ are given by the equation-

$$\frac{a^2\lambda^2}{1-a^2r^2}+\frac{b^2\mu^2}{1-b^2r^2}+\frac{c^2\gamma^2}{1-c^2r^2}=0$$

Answer

(2) b. We have to find the maximum and minimum values of r, where—

$$r^2 = x^2+y^2+z^2 \longrightarrow (1)$$

Also, the variables x, y, z are connected by the relations—

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = (x^2+y^2+z^2)^2 = r^4 \longrightarrow (2)$$

and $\lambda x+\mu y+\gamma z = 0 \longrightarrow (3)$

From (1) —

$$2r\,dr = 2x\,dx+2y\,dy+2z\,dz$$

For a maximum or a minimum of r, we have—

$$dr = 0 \Rightarrow x\,dx+y\,dy+z\,dz = 0 \longrightarrow (4)$$

Differentiating (2), we get

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 4r^3 dr$$

But for a maximum or a minimum of r, we have dr=0

$$\therefore \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \longrightarrow (5)$$

Also, differentiating (3), we get

$$\lambda dx + \mu dy + \nu dz = 0 \longrightarrow (6)$$

Multiplying (4) by 1, (5) by λ_1 and (6) by λ_2 and adding and then equating to zero, the coefficients of dx, dy, dz, we get

$$x + \frac{x}{a^2}\lambda_1 + \lambda\lambda_2 = 0 \longrightarrow (7)$$

$$y + \frac{y}{b^2}\lambda_1 + \mu\lambda_2 = 0 \longrightarrow (8)$$

$$z + \frac{z}{c^2}\lambda_1 + \nu\lambda_2 = 0 \longrightarrow (9)$$

Multiplying (7), (8), (9) by x, y, z respectively and adding, we get

$$r^2 + r^4\lambda_1 + 0\cdot\lambda_2 = 0 \quad \text{or} \quad \lambda_1 = -\frac{1}{r^2}$$

$\therefore$ from (7), we have

$$x - \frac{x}{a^2}\cdot\frac{1}{r^2} + \lambda\lambda_2 = 0$$

$$\text{or} \quad x = \frac{a^2r^2\lambda\lambda_2}{1-a^2r^2}$$

Similarly from (8) & (9), we have

$$y = \frac{b^2r^2\mu\lambda_2}{1-b^2r^2} \quad \& \quad z = \frac{c^2r^2\nu\lambda_2}{1-c^2r^2}$$

Sustituting these values of x, y, z in $\lambda x + \mu y + \nu z$ we get —

$$\frac{a^2 r^2 \lambda^2 \lambda_2}{1-a^2r^2} + \frac{b^2 r^2 \mu^2 \lambda_2}{1-b^2r^2} + \frac{c^2 r^2 \nu^2 \lambda_2}{1-c^2r^2} = 0$$

or $$\frac{a^2\lambda^2}{1-a^2r^2} + \frac{b^2\mu^2}{1-b^2r^2} + \frac{c^2\nu^2}{1-c^2r^2} = 0.$$

(Proved)

Q3 (a) Evaluate $\iint (x+y)^2 dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Answer

3) a). The region of integration can be considered as bounded by —

$y = -b\sqrt{1-x^2/a^2}$, $y = b\sqrt{1-x^2/a^2}$,

$x = -a$ and $x = a$

$$\therefore \iint (x+y)^2 dx\,dy = \int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} (x^2+y^2+2xy)\,dx\,dy$$

, the first integration to be performed w.r.t. y regarding x as constant

$$= \int_{-a}^{a} 2\int_{0}^{b\sqrt{1-x^2/a^2}} (x^2+y^2)\,dx\,dy$$

[$\because$ $2xy$ being an odd function of y, its integration under the given limits of y is 0]

$$= 2\int_{-a}^{a}\left[x^2y + \frac{y^3}{3}\right]_0^{b\sqrt{1-x^2/a^2}}$$

$$= 2\int_{-a}^{a}\left[x^2 b\sqrt{1-\frac{x^2}{a^2}} + \frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[x^2 b\sqrt{1-\frac{x^2}{a^2}} + \frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]dx$$

$$= 4b\int_{0}^{\pi/2}\left[a^2\sin^2\theta\cos\theta + \frac{b^2}{3}\cos^3\theta\right]a\cos\theta\, d\theta$$

, putting $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$= 4ab\int_{0}^{\pi/2}\left[a^2\sin^2\theta\cos^2\theta + \frac{b^2}{3}\cos^4\theta\right]d\theta$$

$$= 4ab\left[a^2\int_{0}^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{b^2}{3}\int_{0}^{\pi/2}\cos^4\theta\, d\theta\right]$$

$$= 4ab\left[a^2\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{b^2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$, by Walli's formula

$$= 4ab\left[\frac{1}{16}\pi a^2 + \frac{1}{16}\pi b^2\right]$$

$$= \frac{1}{4}\pi ab(a^2+b^2)$$

Ans

Q3 (b) Evaluate $\iiint z^2\,dxdydz$ **over the sphere** $x^2+y^2+z^2=1$

Answer

(3)(b). Here the region of integration can be expressed as —

$$-1 \le x \le 1, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$
$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$

∴ the required triple integral

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z^2 \, dx\,dy\,dz$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\frac{z^3}{3}\right]_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \frac{1}{3}\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\,(1-x^2-y^2)^{3/2}\,dy\right] dx$$

$$= \frac{2}{3}\int_{-1}^{1} \left[\int_{-\pi/2}^{\pi/2} \left[(1-x^2)\cos^2\theta\right]^{3/2} \cdot \sqrt{1-x^2}\cos\theta\, d\theta\right] dx,$$

by putting $y = \sqrt{1-x^2}\sin\theta$
$\Rightarrow dy = \sqrt{1-x^2}\cos\theta\, d\theta$
$y = 0 \Rightarrow \theta = 0$ & $y = \sqrt{1-x^2} \Rightarrow \theta = \frac{\pi}{2}$

$$\frac{2}{3}\int_{-1}^{1}\left[2\int_{0}^{\pi/2} (1-x^2)^2 \cos^4\theta\, d\theta\right] dx$$

$$= \frac{4}{3}\int_{-1}^{1} (1-x^2)^2 \, \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\, dx$$

$$= \frac{\pi}{4}\int_{-1}^{1} (1-x^2)^2 dx$$

$$= \frac{\pi}{4}\cdot 2\int_{0}^{1} (1-2x^2+x^4)\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{\pi}{2}\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= \frac{\pi}{2}\cdot\frac{8}{15}$$

$$= \frac{4\pi}{15}$$

Ans

Q4 (a) For what values of η, the equations x + y + z =1, x + 2y + 4z = η, x + 4y + 10z = η^2 have a solution and solve them completely in each case.

Answer

4.) a). The matrix form of the given system is —

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

Performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get —

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta - 1 \\ \eta^2 - 1 \end{bmatrix}$$

Performing $R_3 \to R_3 - 3R_2$, we get —

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta - 1 \\ \eta^2 - 3\eta + 2 \end{bmatrix} \longrightarrow (1)$$

The given equations will be consistent if and only if —

$$\eta^2 - 3\eta + 2 = 0$$

i.e. iff $(\eta - 2)(\eta - 1) = 0$

i.e. iff $\eta = 2, \eta = 1$

Case I). If $\eta = 2$, the eqn (1) becomes —

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The above system of equations is equivalent to

$$x + y + z = 1$$
$$y + 3z = 1$$

$\therefore$ $y = 1 - 3z$, $x = 2z$

Thus $x = 2k$, $y = 1 - 3k$, $z = k$ constitute the general solution where k is an arbitrary constt.

Case II). If $\eta = 1$, the eqn (1) becomes —

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The above system of eqns is equivalent to

$$x + y + z = 1, \quad y + 3z = 0$$

$\therefore$ $y = -3z$, $x = 1 + 2z$

Thus $x = 1 + 2c$, $y = -3c$, $z = c$ constitute the general solution, where c is an arbitrary constant

Ans.

Q4 (b) Determine the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Answer

(4) b). Here —

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic eqn of A is—

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{vmatrix} = 0, \quad C_3 \to C_3 + C_2$$

or $$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

or $(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 1 \end{vmatrix} = 0, \quad R_2 \to R_2 - R_3$

or $(2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right] = 0$

or $(2-\lambda)(\lambda^2 - 10\lambda + 16) = 0$

or $(2-\lambda)(\lambda-2)(\lambda-8) = 0$

∴ the eigenvalues of A are given by—

$\lambda = 2, 2, 8$

The eigenvectors of A corresponding to the eigenvalue 8 are given by the non-zero solutions of the eqn

$(A - 8I)X = 0$

or $\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R_2 \to R_2 - R_1, \; R_3 \to R_3 + R_1$

or $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R_3 \to R_3 - R_2$

The coefficient matrix of these eqns is of rank 2.
∴ these eqns possess $3-2=1$ linearly independent solns

These eqns can be written as—

$-2x_1 - 2x_2 + 2x_3 = 0, \quad -3x_2 - 3x_3 = 0$

The last eqn gives $x_2 = -x_3$

Let us take $x_3 = 1, x_2 = -1$

Then the first eqn gives $x_1 = 2$

$\therefore \; X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue 8.

Every non zero multiple of X_1 is an eigenvector of A corresponding to the eigenvalue 8.

The eigenvectors of A corresponding to the eigenvalue 2 are given by the non-zero solns of the eqn—
$(A - 2I)X = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R_1 \leftrightarrow R_2$$

so

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

The coefficient matrix of these eqns is of rank 1.

$\therefore$ these eqns possess $3 - 1 = 2$ linearly independent solns. These eqns reduce to—

$-2x_1 + x_2 - x_3 = 0$

Hence $X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are two linearly independent solns of this eqn. (6)

$\therefore$ X_2 & X_3 are two eigenvectors of A corresponding to the eigenvalue 2. If c_1 & c_2 are scalars not both equal to zero then $c_1 X_2 + c_2 X_3$ gives all the eigenvectors of A corresponding to the eigenvalue 2.

Ans

Q5 (a) Find by Newton – Raphson method, the real root of the equation:

$$3x = \cos x + 1$$

Nearer to 1, correct to three decimal places.

Answer

5) a). Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -ve$, $f(1) = 3 - 0.5403 - 1$

$= 1.4597 = +ve$

So a root of $f(x) = 0$ lies between 0 and 1. We have to find a root nearer to 1.

$\therefore$ we take $x_0 = 0.6$

Here $f'(x) = 3 + \sin x$

$\therefore$ Newton-Raphson formula gives—

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \longrightarrow (1)$$

Putting $n = 0$, the first approximation x_1 is given by-

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729}$$

$\therefore \quad x_1 = 0.6071$

Putting $n = 1$, the second approximation is—

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049}$$

$x_2 = 0.6071$

Here $x_1 = x_2$

$\therefore$ the desired root is 0.6071.

Ans

Q5 (b) Apply Runge-Kutta method of fourth order to find approximate value of y for x = 0.2, in steps of 0.1, if $\frac{dy}{dx} = x + y^2$, given that y = 1 where x = 0

Answer

(5) b). Given $f(x, y) = x + y^2$

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$k_1 = hf(x_0, y_0) = 0.1\, f(0, 1) = 0.1000$

$k_2 = hf(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1) = 0.1 f(0.05, 1.1)$

$= 0.1152$

$k_3 = hf(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2) = 0.1 f(0.05, 1.1152)$

$= 0.1168$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.1 f(0.1, 1.1168)$

$= 0.1347$

$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347)$

$\therefore k = 0.1165$

which gives $y(0.1) = y_0 + k = 1.1165$

Step II) $x_1 = x_0 + h = 0.1$

$y_1 = 1.1165$, $h = 0.1$

$k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$

$k_2 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) = 0.1f(0.15, 1.1838) = 0.1551$

$k_3 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = 0.1f(0.15, 1.194)$

$= 0.1576$

$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576)$

$= 0.1823$

$\therefore K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.1571$

$\therefore y(0.2) = y_1 + k = 1.2736$

Ans

Q6 (a) Solve the equation cos x dy = y (sin x – y) dx

Answer

5) a). Given –

$\cos x \, dy = y(\sin x - y)\,dx$

$\cos x \frac{dy}{dx} - y\sin x = -y^2$

$\frac{dy}{dx} - y\tan x = -y^2 \sec x$

$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \longrightarrow (1)$

Put $\frac{1}{y} = v$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + v.\tan x = \sec x$, from (1) $\rightarrow$ (2)

$I.F. = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$

Soln of (2) is —

$v.\sec x = \int \sec x.\sec x\,dx + c$

, $c \rightarrow$ arbitrary constant

$v\sec x = \int \sec^2 x\,dx + c$

$v\sec x = \tan x + c$

Put $v = \frac{1}{y}$ —

$\frac{1}{y}\sec x = \tan x + c$

$\therefore \boxed{\sec x = y(c + \tan x)}$

Ans

Q6 (b) Find the orthogonal trajectories of the family of curves:

$$\frac{x^2}{(a^2+\lambda)}+\frac{y^2}{(b^2+\lambda)}=1$$

Where λ is the parameter.

Answer

Q6) b). Given –

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \longrightarrow (1)$$

Differentiating eqn (1) w.r.t. x, we get –

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$$

$$x(b^2+\lambda)+y\left(\frac{dy}{dx}\right)(a^2+\lambda)=0$$

or $$\lambda\left[x+y\left(\frac{dy}{dx}\right)\right]=-\left[b^2x+a^2y\left(\frac{dy}{dx}\right)\right]$$

$$\Rightarrow \lambda=-\frac{\left[b^2x+a^2y\left(\frac{dy}{dx}\right)\right]}{\left[x+y\left(\frac{dy}{dx}\right)\right]}$$

Thus — $a^2+\lambda = \dfrac{(a^2-b^2)x}{x+y\left(\frac{dy}{dx}\right)}$

and $b^2+\lambda = -\dfrac{(a^2-b^2)y\frac{dy}{dx}}{x+y\left(\frac{dy}{dx}\right)}$

Substituting these values in eqn (1), we get

$$\frac{x^2\left[x+y\left(\frac{dy}{dx}\right)\right]}{(a^2-b^2)x} - \frac{y^2\left[x+y\left(\frac{dy}{dx}\right)\right]}{(a^2-b^2)y\frac{dy}{dx}} = 1$$

or $$x^2-y^2+xy\left(\frac{dy}{dx}-\frac{1}{dy/dx}\right) = a^2-b^2 \longrightarrow (2)$$

Put $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in eqn (2), the differential eqn of the orthogonal trajectories is —

$$x^2-y^2+xy\left(-\frac{dx}{dy}+\frac{dy}{dx}\right) = a^2-b^2$$

or $$x^2-y^2+xy\left(\frac{dy}{dx}-\frac{1}{dy/dx}\right) = a^2-b^2$$

which is same as eqn (2).

∴ solving it, we shall get —

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu} = 1$$

μ being the parameter, as the eqn of the orthogonal trajectories. ∴ the system of given confocal conics $\frac{x^2}{(a^2+\lambda)}+\frac{y^2}{(b^2+\lambda)} = 1$ is self-orthogonal.

Ans

Q7 (a) Find the solution of the equation $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$

given that y = 0 and $\frac{dy}{dx}$ = 2 when x = 0

Answer

(7) a) The given differential eqn can be written

$$(D^2+4)y = 8\cos 2x \quad \longrightarrow (1)$$

The A.E. is —

$$m^2+4=0 \Rightarrow m = \pm 2i$$

$\therefore$ C.F. $= c_1\cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4} 8\cos 2x = 8 \frac{1}{D^2+4}\cos 2x$$

$$= 8\left[\text{the real part in } \frac{1}{D^2+4} e^{2ix}\right] \longrightarrow (2)$$

Now—

$$\frac{1}{D^2+4} e^{2ix} = \frac{1}{(D+2i)(D-2i)} e^{2ix}$$

$$= \frac{1}{(2i+2i)(D-2i)} e^{2ix}$$

, putting $2i$ for D in the factor $D+2i$

$$= \frac{1}{4i(D-2i)} e^{2ix}$$

$$= \frac{1}{4i} e^{2ix} \frac{1}{(D+2i-2i)} 1$$

$$= \frac{1}{4i} e^{2ix} \frac{1}{D} 1$$

$$= \frac{1}{4i} e^{2ix} x$$

$$= \frac{x}{4i}(\cos 2x + i\sin 2x)$$

$$= -\frac{ix}{4}(\cos 2x + i\sin 2x)$$

(9)

$= -\frac{ix}{4}\cos 2x + \frac{x}{4}\sin 2x$

$\therefore$ the real part in $\frac{1}{D^2+4} e^{2ix}$

$= \frac{x}{4}\sin 2x$

Put in eqn (2), we get—

P.I. $= 8 \times \frac{x}{4}\sin 2x = 2x\sin 2x$

$\therefore$ the general solution of the given diff. eqn is —

$y = c_1\cos 2x + c_2\sin 2x + 2x\sin 2x \longrightarrow (3)$

Given $y = 0$ and $\frac{dy}{dx} = 2$ when $x = 0$

From (3) —

$\frac{dy}{dx} = -2c_1\sin 2x + 2c_2\cos 2x + 2\sin 2x + 4x\cos 2x \longrightarrow (4)$

Putting given conditions in (3) & (4), we get—

$0 = c_1$ and $2 = 2c_2$
$\Rightarrow c_2 = 1$

Putting values of c_1 & c_2 in (3), we get—

$\boxed{y = \sin 2x + 2x\sin 2x}$

which is the required solution.

Ans

Q7 (b) Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Answer

(7) b). Given diff. eqn is—

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \longrightarrow (1)$$

C.F. of the eqn (1) is given by—

$$\frac{d^2y}{dx^2} - y = 0$$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x}$$

Let $y = Ae^x + Be^{-x}$ be the complete soln of the given diff. eqn, where A & B are functions of x so chosen that the given diff. eqn will be satisfied.

$$\therefore \frac{dy}{dx} = Ae^x - Be^{-x} + \frac{dA}{dx}e^x + \frac{dB}{dx}e^{-x}$$

Let us choose A & B s.t. —

$$e^x \frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \longrightarrow (2)$$

$$\therefore \frac{dy}{dx} = Ae^x - Be^{-x}$$

and $\frac{d^2y}{dx^2} = \frac{dA}{dx}e^x - \frac{dB}{dx}e^{-x} + Ae^x + Be^{-x}$

Putting these values in the given diff. eqn (1), we get —

$$e^x \frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{1+e^x} \longrightarrow (3)$$

Solving (2) & (3), we get —

$$\frac{dA}{dx} = \frac{e^{-x}}{1+e^x} \quad \text{and} \quad \frac{dB}{dx} = -\frac{e^x}{1+e^x}$$

Integrating these, we get —

$$A = \int \frac{e^{-x}}{1+e^x}\,dx + c_1$$

$$= \int \frac{dz}{z^2(1+z)} + c_1, \quad \text{putting } e^x = z$$

$$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right) dz + c_1$$

$$= -\frac{1}{z} - \log z + \log(1+z) + c_1$$

$$= -\frac{1}{z} + \log\left(\frac{1+z}{z}\right) + c_1$$

$$A = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x} + c_1$$

and $B = -\int \frac{e^x}{1+e^x}\,dx + c_2$

$$= -\log(1+e^x) + c_2$$

Putting these values in $y = Ae^x + Be^{-x}$, the general solution of the given diff. eqn is—

$$y = c_1 e^x + c_2 e^{-x} + e^x \log\left(\frac{1+e^x}{e^x}\right) - 1 - e^{-x}\log(1+e^x)$$

Ans

Q8 (a) Show that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx = \frac{1}{(a+b)^m a^n}B(m,n)$

where B (m, n) is Beta function.

Answer

Q8) a) The given integral is—

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1}\left(\frac{1-x}{a+bx}\right)^{n-1}\frac{1}{(a+bx)^2}dx$$

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b}$

so that $\dfrac{(a+bx)\cdot 1 - x\cdot b}{(a+bx)^2}dx = \dfrac{dy}{a+b}$

i.e. $\dfrac{1}{(a+bx)^2}dx = \dfrac{dy}{a(a+b)}$

Further —

$$\frac{1-x}{a+bx} = \frac{1}{a}\,\frac{a-ax}{a+bx} = \frac{1}{a}\left[\frac{a+bx-ax-bx}{a+bx}\right]$$

$$= \frac{1}{a}\left[1 - \frac{x(a+b)}{a+bx}\right]$$

$$= \frac{1-y}{a}$$

Also, when $x=0,\ y=0$ and when $x=1,\ y=1$

$$\therefore I = \int_0^1\left(\frac{y}{a+b}\right)^{m-1}\left(\frac{1-y}{a}\right)^{n-1}\frac{dy}{a(a+b)}$$

$$= \frac{1}{(a+b)^m a^n}\int_0^1 y^{m-1}(1-y)^{n-1}dy = \frac{B(m,n)}{(a+b)^m a^n}$$

(Proved)

Q8 (b) Solve following differential equation $\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2 + 2x + 2$ in power of x.

Answer

8) b) Given diff. eqn is—

$$\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2+2x+2 \quad \longrightarrow (1)$$

Here $x=0$ is an ordinary point.
Let a trial solution in the form of the series of the given diff. eqn be—

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots \quad \longrightarrow (2)$$
$$= \sum_{n=0}^{\infty} c_n x^n$$

Differentiating (1), we get—

$$\frac{dy}{dx} = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} + \cdots \quad \longrightarrow (3)$$

and $$\frac{d^2y}{dx^2} = 2c_2 + 6c_3x + \cdots + n(n-1)c_nx^{n-2} + \cdots \quad \longrightarrow (4)$$

Putting these values in the given eqn (1), we get—

$$(2c_2 + 6c_3x + \cdots) - 2x^2(c_1 + 2c_2x + 3c_3x^2 + \cdots) + 4x(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots) - x^2 - 2x - 2 = 0$$

or $$(2c_2 - 2) + (6c_3 + 4c_0 - 2)x + (12c_4 + 2c_1 - 1)x^2 + 20c_5x^3 + \cdots + \left[(n+2)(n+1)c_{n+2} - 2(n-1)c_{n-1} + 4c_{n-1}\right]x^n + \cdots = 0$$

which is an identity in x.
We can equate to zero the coefficients of various powers of x,

Equating to zero, the coefficients of various powers of x, we get—

$2c_2 - 2 = 0$ i.e. $c_2 = 1$

$6c_3 + 4c_0 - 2 = 0 \Rightarrow c_3 = \frac{1}{3} - \frac{2}{3}c_0$

$12c_4 + 2c_1 - 1 = 0 \Rightarrow c_4 = \frac{1}{12} - \frac{1}{6}c_1$

All other coefficients are given by the relation-

$$c_{n+2} = \frac{2(n-3)}{(n+1)(n+2)} c_{n-1}, \quad n \geq 3$$

Hence, the required complete solution in series is—

$$y = c_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 - \dots\right) + c_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - \dots\right) + x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots$$

where c_0 and c_1 are arbitrary constants.

Ans

Q9 (a) Prove that:

$$x^2 J_n''(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x)$$

Answer

9) a). We shall use recurrence formula –

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \rightarrow (1)$$

Differentiating (1) w.r.t. 'x', we get –

$$x J_n''(x) + J_n'(x) = n J_n'(x) - x J_{n+1}'(x) - J_{n+1}(x)$$

$$x J_n''(x) = (n-1) J_n'(x) - x J_{n+1}'(x) - J_{n+1}(x)$$

or $$x^2 J_n''(x) = (n-1) x J_n'(x) - x^2 J_{n+1}'(x) - x J_{n+1}(x) \quad \rightarrow (2)$$

Again using recurrence formula –

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

writing $(n+1)$ for n, we get –

$$x J_{n+1}'(x) = -(n+1) J_{n+1}(x) + x J_n(x) \quad \rightarrow (3)$$

Putting values of $x J_n'(x)$ from (1) & of $x J_{n+1}'(x)$ from (3) in (2), we get –

$$x^2 J_n''(x) = (n-1)[n J_n(x) - x J_{n+1}(x)] - x[-(n+1) J_{n+1}(x) + x J_n(x)] - x J_{n+1}(x)$$

or $$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x)$$

(Proved)

Q9 (b) Prove that

$$\int_{-1}^{+1} (1 - x^2) P'_m P'_n \, dx = 0$$

Where m and n are distinct positive integers.

Answer

(9) b). We have—

$$\int_{-1}^{+1} (1-x^2) P_m' P_n' \, dx$$

$$= \Big[(1-x^2) P_m' P_n\Big]_{-1}^{+1} - \int_{-1}^{+1} \Big[P_n \frac{d}{dx}\{(1-x^2) P_m'\}\Big] dx$$

, integrating by parts taking P_n' as the second function

$$= -\int_{-1}^{+1} \Big[P_n \frac{d}{dx}\{(1-x^2) P_m'\}\Big] dx \longrightarrow (1)$$

Now, since P_m is a solution of the Legendre's eqn, therefore—

$$(1-x^2) P_m'' - 2x P_m' + m(m+1) P_m = 0$$

or $$\frac{d}{dx}\Big[(1-x^2) P_m'\Big] = -m(m+1) P_m$$

Putting this value in eqn (1), we get—

$$\int_{-1}^{+1} (1-x^2) P_m' P_n' \, dx = -\int_{-1}^{+1} \Big[-P_n \, m(m+1) P_m\Big] dx$$

$$= m(m+1) \int_{-1}^{+1} P_n P_m \, dx$$

$$= 0, \quad \text{since } m \neq n$$

(Proved).

Text Books

1. Higher Engineering Mathematics, Dr. B.S. Grewal, 40th edition 2007, Khanna Publishers, Delhi.

2. Text book of Engineering Mathematics, N.P. Bali & Manish Goyal, 7th edition 2007, Laxmi Publication (P) Ltd.